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A COMPARISON OF SOME PROPOSALS FOR THE MIXING LENGTH NEAR A WALL

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NOMENCLATURE

A_+, function of p_+ and v_{w+};
c_f, coefficient of friction for pipe flow $2\tau_w/\rho u_G^2$;
H, shape factor of boundary layer;
K, acceleration parameter $(\nu/u_G^2)\,\mathrm{d}u_G/\mathrm{d}x$;
R_2, momentum thickness Reynolds number;
p_+, dimensionless pressure gradient $-\dfrac{\nu}{\tau_w\sqrt{(\tau_w/\rho)}}\dfrac{\mathrm{d}p}{\mathrm{d}x}$;
Re, Reynolds number based on hydraulic diameter and bulk velocity;
u_G, free stream velocity;
v_w, velocity of suction through surface;
x, distance downstream;
y, distance normal to wall;
ν, kinematic viscosity;
ρ, fluid density;
τ, shear stress.

Subscripts
$+$, quantity non-dimensionalised by $\sqrt{(\tau_w/\rho)}$ and ν;
w, wall value.

INTRODUCTION

MOST users of Prandtl's mixing-length hypothesis have adopted a version of Van Driest's [1] proposal for the viscous damping of mixing length, l, near a wall:

$$l = \varkappa y(1 - \exp - D) \tag{1}$$

where $\varkappa$ is a constant and D is a function, *inter alia* of distance from the wall, which we term the damping function. Van Driest originally suggested that D should vary linearly with y_+ a form which predicted well the near-wall velocity profile of Laufer [2].

With new computational techniques, more complex turbulent flow could be examined and the limitations of Van Driest's proposal became evident. Stanton numbers calculated for severely accelerated flows were too high by 80 per cent or more, and there were similar shortcomings in calculations of transpired boundary layers. Several workers have therefore proposed variants of Van Driest's which are claimed to possess wider validity than the original. We here report calculations of four test flows employing Van Driest's version of D and nine variants of it.

Of the forms considered, listed in Table 1, perhaps the most popular is entry 2 which replaces τ_w by τ, a reasonable generalization of Van Driest's formula. It has been reported however that measurements are still not well predicted when pressure gradients or transpiration rates are large. Several

Table 1. The damping functions examined

Number	*Originator(s)*	*Formula for D*
1	Van Driest [1]	$y_+/26$
2	Patankar and Spalding [4]	$y_+(\tau_+)^{\frac{1}{2}}/26$
3	Cebeci [5]	$\left[\dfrac{p_+}{v_{w+}}(1-E)+E\right]^{\frac{1}{2}} y_+/26$ where $E \equiv \exp(11{\cdot}8\,v_{w+})$
4	Loyd *et al.* [6]	Tabular values of A_+ as function of v_{w+}, p_+. Then $D = y_+\tau_+^{\frac{1}{2}}/A_+$
5	Powell and Strong [7]	$A_+ = 26(1 + v_{w+}u_+)^{1{\cdot}4}$ Then: $D = y_+\tau_+^{\frac{1}{2}}\dfrac{v_{w+}}{2}\{1 \pm \tfrac{1}{2}[+ (1 + 64/(A_+m_+)^4)^{\frac{1}{2}}]^{\frac{1}{2}}\}$
6	Rotta [8]	$y_+/(13{\cdot}6 + 12{\cdot}4\exp 10{\cdot}75v_{w+})$
7	Baker *et al.* [11]	$y_+\tau_+/26$
8	present work	$y_+\tau_+^{\frac{3}{2}}/26$
9	present work	$y_+\tau_+^2/26$
10	Spalding [13]	$\int_0^{y_+}\tau_+\,\mathrm{d}y_+/26$

workers have employed correlations of m_+ and p_+ to remedy these shortcomings e.g. versions 3–6. The authors and their colleagues prefer instead to express the above influence through the shear-stress ratio, τ_+. This approach has at least two advantages. First, it avoids the too rapid streamwise variation [9, 10] in D which occurs when D depends on the local value of p_+. Second, expressions based on τ_+ provide correct trends when shear stress variations arise from other causes e.g. buoyancy. Version 7, a simplified form of [12], has been previously used to obtain extensive predictions of blown boundary layers in pressure gradients, including density gradients. Versions 8 and 9 are similar to 7 but employ larger exponents of τ_+. The last entry was devised to account for τ_+ variations when used in "wall functions" of the Patankar–Spalding finite-difference procedure [4].

Attention was confined to momentum-transport processes because to have included heat-transfer predictions would have entangled the distribution of turbulent Prandtl number with that of l. The following four test cases were chosen because predictions of them were very sensitive to the form of D:

(i) the asymptotic sink flow
(ii) low-Reynolds-number turbulent channel flow
(iii) low-Reynolds-number turbulent pipe flow
(iv) the asymptotic suction boundary layer.

Predictions are compared with data and with predictions obtained with a two-equation turbulence model [15], except for case (iv) where comparison is made only with the 2-equation model [16] predictions because no data of sucked boundary layers seemed close enough to the asymptotic state. The predictions were made with a version of the Patankar–Spalding program [4] with negligible turbulent stress at the near-wall node and with the effective kinematic viscosity equal to:

$$\nu + l^2\left|\frac{\partial u}{\partial y}\right|. \qquad (2)$$

Near the wall l was given by (1) and $\varkappa$ assigned the value 0·41 for cases (i) and (iv) and 0·40 for cases (ii) and (iii). However, for all y greater than at which l given by (1) first exceeds λy_G, l was assigned this maximum value. For the external flows, λ was set to 0·075 and for the confined flows to 0·11; these choices were made on the basis of *high* Reynolds number calculations of the same flows. In the flows tested, however, the value of λ has only secondary influence on mean-flow parameters.

COMPARISON AND ASSESSMENT

In a sink-flow boundary layer K is constant for all x; it thus follows from the momentum equation that R_2 will tend to a constant value. The flow structure is very sensitive to the value of K: for values between $0·5 \times 10^{-6}$ and $3·0 \times 10^{-6}$ the sublayer is thicker than at high Reynolds numbers; for much steeper accelerations only laminar flow can persist. Differences between the values of R_2 predicted by the various damping functions increase markedly with K (Fig. 1). Predictions with Van Driest's version lie well above the measurements while formulae 4, 8 and 9 give the best agreement.

Figures 2 and 3 compare experimental and predicted friction factors for plane channel and pipe flow. We consider Driest's proposal. For both flows the best predictions are only the regime where c_f decreases as Re increases. For both pipe and channel flow most of the damping functions predict too large c_f's, though all variants improve on Van

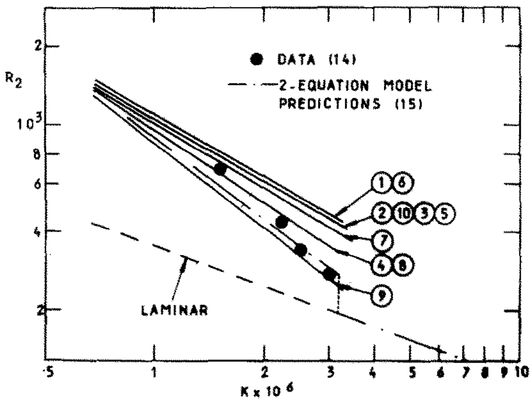

FIG. 1. Variation of R_2 with K for sink flows.

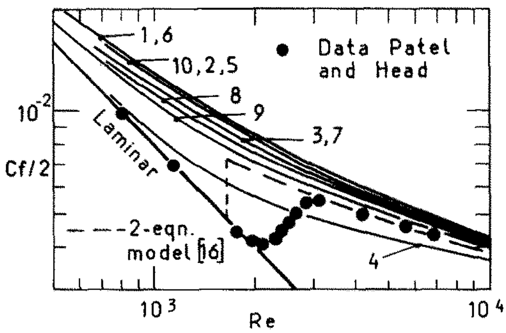

FIG. 2. Friction factor in low Re pipe flow.

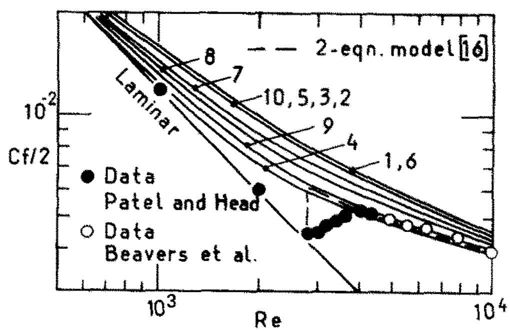

FIG. 3. Friction factor in low Re channel flow.

Driests' proposal. For both flows the best predictions are given by formulae 4 and 9; the next best versions are 8 and 7. The 2-equation model gives, for both cases, predictions in somewhat closer agreement with measurements than any of the mixing-length models.

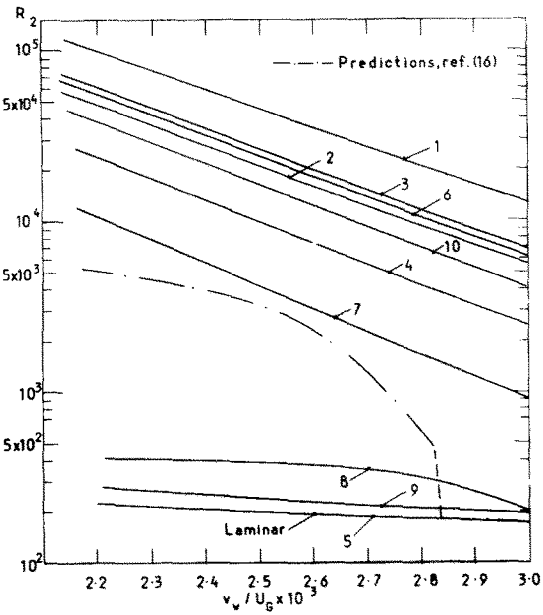

FIG. 4. Asymptotic suction boundary layer.

Finally, Fig. 4 shows the asymptotic momentum-thickness Reynolds numbers for sucked boundary layers vs. v_w/u_G. There is a startling variation among the predictions: for $(v_w/u_G) = 0{\cdot}00282$ Van Driest's proposal leads to an R_2 of 19730 while formula 5 gives a value of only 183, a 100:1 variation. No damping function leads to predicted R_2's in very close agreement with those of the 2-equation model (which in the absence of other evidence provides the target values for this flow). Over the range of suction rates considered, versions 7 and 8 are in closest agreement; these are followed by versions 5 and 9.

CONCLUSIONS

1. When in turbulent flow past a smooth surface, there are steep negative gradients of shear stress normal to the wall, the Van Driest mixing-length proposal gives consistently too large values of effective viscosity. Of the nine variants tested, versions 4, 8 and 9 gave the best overall agreement. The latter two versions of the damping function possess the simple form: $D = y_+ \tau_+^n/26$; the optimum value of the exponent n is about 1·7.
2. Although no mixing-length model gave as good predictions as did the two-equation model [15], the former requires only 20 per cent of the computational time of the latter. Thus, for certain applications the recommended versions of the mixing-length hypothesis probably offer the best kind of turbulence model for wall boundary-layer calculations.

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